

On a generalized Sturm theorem

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May 21, 2009

Abstract

Sturm oscillation theorem for second order differential equations was generalized to systems and higher order equations with positive leading coefficient by several authors. What we propose here is a Sturm theorem for indefinite systems with Dirichlet boundary conditions of the form

$$p_{2m} \frac{d^{2m}u}{dx^{2m}} + p_{2m-2}(x) \frac{d^{2m-2}u}{dx^{2m-2}} + \cdots + p_1(x) \frac{du}{dx} + p_0(x)u = 0,$$

where p_i is a smooth path of matrices on the complex n -dimensional vector space \mathbb{C}^n and p_{2m} is the symmetry represented by $\text{diag}(I_{n-\nu}, -I_\nu)$ for some integer $0 \leq \nu \leq n$.

1 Introduction

Sturm oscillation theorem deals with differential equations of the form

$$-(pu')' + qu = 0 \tag{1.1}$$

where $'$ denotes differentiation and where p and q are given (differentiable) functions with $p > 0$. Let $V[0, 1]$ be the vector space $C_0^\infty([0, 1])$ and, for each $x \in [0, 1]$, let us consider the following two quadratic forms on $V[0, x]$ and $V[0, 1]$ respectively given by $q_x(u) := \int_0^x \Omega[u]dx$ and $q_\lambda(u) := \int_0^1 (\Omega[u] - \lambda u^2)dx$, where $\Omega[u] = p|u'|^2 + q|u|^2$. Then, the Sturm oscillation theorem can be restated as follows:

$$\sum_{0 < x < 1} \dim \ker q_x(u) = \sum_{\lambda < 0} \dim \ker q_\lambda(u). \tag{1.2}$$

In this paper we shall study the Dirichlet boundary value problem for the linear differential equation:

$$l(x, D)u := p_{2m} \frac{d^{2m}u}{dx^{2m}} + p_{2m-2}(x) \frac{d^{2m-2}u}{dx^{2m-2}} + \cdots + p_1(x) \frac{du}{dx} + p_0(x)u = 0, \quad x \in [0, 1] \tag{1.3}$$

where p_i is a smooth path of matrices on the complex n -dimensional vector space \mathbb{C}^n and p_{2m} is the symmetry represented by the diagonal block matrix $\text{diag}(I_{n-\nu}, -I_\nu)$ for some integer $0 \leq \nu \leq n$, by using the reformulation in terms of calculus of variations as in [3]. However, we observe in this respect, that the situation we are dealing with is completely different and a new approach is needed, since both the hand-sides of the equation (1.2) are meaningless. What we propose here is a different definition of both sides of formula (1.2). In fact, in our case, the Morse index is not well-defined and the natural substitute for the right hand-side will be the *spectral flow* of a suitable family of Fredholm Hermitian forms. Furthermore, the left hand side will be replaced by a Maslov-type index obtained by specifying a suitable intersection theory in the classical \mathcal{U} -manifolds context.

It is worth noticing a generalization of the Sturm oscillation theorem in an indefinite situation

*The author was partially supported by MIUR project *Variational Methods and Nonlinear Differential Equations*.

recently obtained in [10]. Here the author firstly reduces the even order differential system to a first order Hamiltonian system and then proves the equality between the spectral flow of a path of unbounded self-adjoint Fredholm operator and the Maslov index of a suitable path of Lagrangian subspaces.

2 Linear preliminaries

Spectral flow for Fredholm Hermitian forms. Let H be a complex separable Hilbert space. A bounded self-adjoint operator $A : H \rightarrow H$ is Fredholm if $\ker A$ is finite dimensional its image is closed and $\operatorname{coker} A$ is also finite dimensional. The topological group $\operatorname{Gl}(H)$ of all automorphisms of H acts naturally on the space of all self-adjoint Fredholm operators $\Phi_S(H)$ by cogredience sending $A \in \Phi_S(H)$ to S^*AS . This induces an action of paths in $\operatorname{Gl}(H)$ on paths in $\Phi_S(H)$. As in the real case, for any path $A : [a, b] \rightarrow \Phi_S(H)$ there exist a path $M : [a, b] \rightarrow \operatorname{Gl}(H)$ and a symmetry \mathcal{J} (i.e. a bounded linear operator such that $\mathcal{J}^2 = \operatorname{Id}$) such that $M^*(t)A(t)M(t) = \mathcal{J} + K(t)$ with $K(t)$ compact for each $t \in [a, b]$. Assuming that the path A has invertible endpoints and denoting by μ_{rel} the *relative Morse index*, then the *spectral flow of the path A* is the integer

$$\operatorname{sf}(A, [a, b]) \equiv \mu_{\text{rel}}(\mathcal{J} + K(a), \mathcal{J} + K(b)),$$

where $\mathcal{J} + K$ is any compact perturbation of a symmetry cogredient with A . By the properties of the relative Morse index it is easy to check that this number is well-defined. The spectral flow $\operatorname{sf}(A, [a, b])$ is additive and invariant under homotopies with invertible end points. We refer to [5] for further details.

A *Fredholm Hermitian form* on H is a function $q : H \rightarrow \mathbb{R}$ such that there exists a bounded symmetric sesquilinear form $b = b_q : H \times H \rightarrow \mathbb{C}$ with $q(u) = b(u, u)$ and with $\ker b$ of finite dimension. We denote by $\operatorname{Herm}_F(H)$ the set of all Fredholm Hermitian forms. It is possible to prove that $\operatorname{Herm}_F(H)$ is an open subset (in the operator norm topology) of $\operatorname{Herm}(H)$ which is stable under perturbations by weakly continuous Hermitian forms. Moreover a Hermitian form is called *non degenerate* if the map $u \rightarrow b_q(u, -)$ is an isomorphism between H and its dual H^* . Furthermore a path of Fredholm Hermitian forms $q : [a, b] \rightarrow \operatorname{Herm}_F(H)$ with non degenerate end points $q(a)$ and $q(b)$ will be called *admissible*.

DEFINITION 2.1 *The spectral flow of an admissible path $q : [a, b] \rightarrow \operatorname{Herm}_F(H)$ is given by*

$$\operatorname{sf}(q, [a, b]) := \operatorname{sf}(A_q, [a, b])$$

where $A_{q(t)}$ is the unique self-adjoint Fredholm operator such that $\langle A_{q(t)}u, u \rangle = q(t)(u)$, for all $u \in H$.

As consequence of the invariance of the spectral flow under cogredience, it can be proved that this is independent from the choice of the scalar product. Now given any differentiable path $q : [a, b] \rightarrow \operatorname{Herm}_F(H)$ at the point t , then the derivative $\dot{q}(t)$ is also a Fredholm Hermitian form. We will say that a point t is a *crossing point* if $\ker b_{q(t)} \neq \{0\}$, and we will say that the crossing point t is *regular* if the *crossing form* $\Gamma(q, t)$, defined as the restriction of the derivative $\dot{q}(t)$ to the subspace $\ker b_{q(t)}$, is nondegenerate. It is possible to prove that regular crossings are isolated and that the property of having only regular crossings is generic for paths in $\operatorname{Herm}_F(H)$.

PROPOSITION 2.2 *If all crossing points of the path are regular then they are in a finite number and $\operatorname{sf}(q, [a, b]) = \sum_i \operatorname{sign} \Gamma(q, t_i)$.*

The structure of \mathcal{U} -manifolds and the EM-index. In this paragraph we will briefly recall some useful facts about the \mathcal{U} -manifold and we will define the *EM-index*, by generalizing the intersection theory proposed by Edwards in [3].

NOTATION 2.3 *A (real-valued) Hermitian form on a complex vector space V is a real valued function Q on V which satisfies:*

- (i) The parallelogram law: $Q[v_1 + v_2] - Q[v_1 - v_2] = 2(Q[v_1] + Q[v_2])$ for all $v_1, v_2 \in V$;
- (ii) $Q[cv] = |c|^2 Q[v]$ for all $c \in \mathbb{C}$ and $v \in V$.

Such a function is of the form $Q[v] = Q[v, v]$ where $Q : V \times V \rightarrow \mathbb{C}$ is a uniquely determined symmetric sesquilinear form.

DEFINITION 2.4 A superhermitian space is a pair (S, h) , where

- (i) S is a complex even dimensional vector space;
- (ii) h is a non degenerate Hermitian form of zero signature, called superhermitian structure.

We term *superlagrangian* subspace any subspace L of the superhermitian space (S, h) of dimension $1/2 \dim S$ on which the superhermitian structure h vanishes identically and we will refer with the name of \mathcal{U} -manifold to the set $\mathcal{U}(S, h)$ of all superlagrangian subspaces L of (S, h) . We observe that from the topological viewpoint the \mathcal{U} -manifold is homeomorphic to the unitary group $U(n)$, where $n = \dim S/2$. We will refer to [3, Section 4], for further details.

Now, given a finite dimensional complex vector space V , let us consider the space $S := V \oplus V^*$. If $\zeta := (\xi, \eta) \in S$ and if \Im denotes the imaginary part of a complex number, S has a naturally associated superhermitian structure given by $h[\zeta] := \Im \langle \xi, \eta \rangle$ and usually called *standard superhermitian structure*. Every superlagrangian subspace P_0 determines a decomposition of the space of all superlagrangian subspaces as a disjoint union

$$\mathcal{U} = \bigcup_{k=0}^n \mathcal{U}_k(P_0),$$

where $n = \dim V$ and where, for each k , $\mathcal{U}_k(P_0)$ is the submanifold of those superlagrangian subspaces which intersect P_0 in a subspace of dimension k ; i.e.

$$\mathcal{U}_k(P_0) := \{P \in \mathcal{U} : \dim(P \cap P_0) = k\}.$$

We define the following variety

$$\mathcal{S}(P_0) = \bigcup_{k=1}^n \mathcal{U}_k(P_0).$$

DEFINITION 2.5 Given a pair $P_0, P_1 \in \mathcal{U}(S, h)$ of complementary superlagrangians and identifying P_0^* with P_1 via the symmetric sesquilinear form $\langle \cdot, \cdot \rangle$, we can define the Hermitian form $\varphi_{P_0, P_1} : \mathcal{U}_0(P_1) \rightarrow \text{Herm}(P_0)$ as

$$\varphi_{P_0, P_1}(P) : P_0 \oplus P_1 \rightarrow \mathbb{R} : (u, T_P u) \mapsto \Im \langle u, -iT_P u \rangle$$

where $T_P : P_0 \rightarrow P_1$ is the unique Hermitian operator whose graph is P .

Otherwise, the Hermitian form φ_{P_0, P_1} on P can be defined in the following way. Let $j : S \rightarrow S$ be the unique map which is the identity on P_0 and the multiplication by $-i$ on P_1 . Then it can be easily checked that

$$\varphi_{P_0, P_1}(P)(v) = h[jv] \quad \forall v \in P, \quad (2.1)$$

where h is the standard superhermitian structure.¹

Given any $P \in \mathcal{U}$, it is possible to define a canonical isomorphism from $T_P \mathcal{U}$ and $\text{Herm}(P)$. In fact, let $P_0, P_1 \in \mathcal{U}$ be a pair of complementary superlagrangians and let φ_{P_0, P_1} be a chart of \mathcal{U} . Then the differential $d : T_P \mathcal{U}_0(P_1) \rightarrow \text{Herm}(P)$ is the map which send a point $\hat{P} \in T_P \mathcal{U}_0(P_1)$

¹We observe that, formula (2.1) is the definition of the non-trivial invariant α defined by Edwards in [3, Section 4] on triples of superlagrangian subspaces.

into the Hermitian form $Q(P, \hat{P})$ on P defined as follows. For all $\varepsilon > 0$ sufficiently small, let us consider the curve $(-\varepsilon, \varepsilon) \ni t \mapsto P(t) \in \mathcal{U}$ such that $P(0) = P$ and $P'(0) = \hat{P}$. Then

$$Q(P, \hat{P})(v) := \left. \frac{d}{dt} \right|_{t=0} \Im \langle u, -iT_{P(t)}u \rangle = \left. \frac{d}{dt} \right|_{t=0} h[jv(t)] \quad (2.2)$$

for $v(t) := (u, T_{P(t)}u)$, $v := v(0) \in P$ and where $t \mapsto T_{P(t)}$ is the path of Hermitian operators contained in the domain of the chart and such that their graphs agrees with the path of superlagrangian subspaces $t \mapsto P(t)$ in a sufficiently small neighborhood of $t = 0$. The differential of the chart gives an isomorphism between $T_P \mathcal{U}_0(P_1)$ and $\text{Herm}(P)$ and an easy computation shows that such isomorphism does not depend on the choice of P_1 . Summing up, the following result holds.

PROPOSITION 2.6 *The \mathcal{U} -manifold is a regular algebraic variety of (complex) dimension $\frac{n(n+1)}{2}$. Moreover $(\mathcal{U}_0(P_1), \varphi_{P_0, P_1})$, when (P_0, P_1) runs in the set of all pairs of complementary superlagrangians form an atlas of \mathcal{U} . The differential of $\varphi_{P_0, P_1}(P)$ at P does not depend on the choice of $P_1 \in \mathcal{U}_0(P_0)$ and therefore defines a canonical identification of $T_P \mathcal{U}$ with $\text{Herm}(P)$.*

Given any differentiable path $p: [a, b] \rightarrow \mathcal{U}$, we say that p has a *crossing* with $\mathcal{S}(P_0)$ at the instant $t = t_0$ if $p(t_0) \in \mathcal{S}(P_0)$. At each non transverse crossing time $t_0 \in [a, b]$ we define the *crossing form*

$$\Gamma(p, P_0, t_0) = Q(p(t_0), p'(t_0))|_{p(t_0) \cap P_0} \quad (2.3)$$

and we say that a crossing t is called *regular* if the crossing form $\Gamma(p, P_0, t_0)$ is nonsingular. It is easy to prove that regular crossings are isolated and therefore on a compact interval are in a finite number. Moreover if $p(a), p(b) \notin \mathcal{S}(P_0)$ then p is said an *admissible path*.

REMARK 2.7 *We observe that given a superhermitian space (S, h) it can be shown that the pair defined by (\tilde{S}, \tilde{h}) where $\tilde{S} := S \oplus S$ and $\tilde{h} := -h \oplus h$ is a superhermitian space. With this respect and by defining \tilde{j} as $j \oplus j$, then the crossing form can be formally represented by formulas (2.2)-(2.3) simply by writing \tilde{h} instead of h and \tilde{j} instead of j .*

THEOREM 2.8 *Fix $P_0 \in \mathcal{U}$. Then there exists one and only one map*

$$\mu_{\text{EM}}(\cdot, P_0) : C^0([a, b], \mathcal{U}) \longrightarrow \mathbb{Z}$$

satisfying the following axioms:

- (i) (Homotopy invariance) *If $p_0, p_1: [a, b] \rightarrow \mathcal{U}$ are two homotopic curves of superlagrangian subspaces with $p(a), p(b) \notin \mathcal{S}(P_0)$ then they have the same EM-index.*
- (ii) (Catenation) *For $a < c < b$, if $p(c) \notin \mathcal{S}(P_0)$, then*

$$\mu_{\text{EM}}(p, P_0) = \mu_{\text{EM}}(p|_{[a, c]}, P_0) + \mu_{\text{EM}}(p|_{[c, b]}, P_0).$$

- (iii) (Localization) *If $P_0 := \mathbb{C}^n \times \{0\}$ and $p(t) := \text{Graph}(H(t))$ where $t \mapsto H(t)$ is an admissible path of Hermitian matrices having only a regular crossing at $t = t_0$, then we have*

$$\mu_{\text{EM}}(p, P_0) = \frac{1}{2} \text{sign } H(t_0 + \varepsilon) - \frac{1}{2} \text{sign } H(t_0 - \varepsilon),$$

where ε is any positive real number.

The integer $\mu_{\text{EM}}(p, P_0)$ is called the Edwards-Maslov index of P_0 or briefly EM-index.

Proof. Observe that Axioms (i)–(ii) say that the Maslov index is an homomorphism of the relative homotopy group $\pi_1(\mathcal{U}, \mathcal{U} \setminus \mathcal{S}(P_0))$ into the integers \mathbb{Z} . Now, since \mathcal{U} is homeomorphic to the unitary group and since $\mathcal{U} \setminus \mathcal{S}(P_0)$ is a cell, by excision axiom we have that $\pi_1(\mathcal{U}, \mathcal{U} \setminus \mathcal{S}(P_0)) \cong \mathbb{Z}$. The localization axiom will determine this homomorphism uniquely. It remains only to show that

any two curves of the type described by axiom (iii) are in the same relative homotopy class. To do so, let p_1, p_2 be two such curves. By using Kato's selection theorem it is not restrictive to assume that this two curves are of the form

$$p_j(t) = \Delta(-1, -1, \dots, -1, t, 1, 1, \dots, 1) \quad t \in [-1, 1], \text{ and } j = 1, 2,$$

where Δ denotes the diagonal matrix. Now the thesis follows by the definition of Γ at t_0 and by taking into account that

$$\text{sign } H(t_0 \pm \varepsilon) = \text{sign } H(t_0) \pm \text{sign } \Gamma(p, P_0, t_0).$$

□

Since regular crossing are isolated then on a compact interval are in a finite number and the following result holds.

PROPOSITION 2.9 *For an admissible differentiable path $p : [a, b] \rightarrow \mathcal{U}$ having only regular crossings, we have:*

$$\mu_{\text{EM}}(p, P_0) = \sum_{t_0 \in (a, b)} \text{sign } \Gamma(p, P_0, t_0) \quad (2.4)$$

where we denote by sign the signature of a Hermitian form and where the summation runs over all crossings t .

Proof. The proof of this formula follows by local chart computation obtained by using formulas (2.2)- (2.3) and the localization and concatenation properties.

REMARK 2.10 *We observe that in the case of positive definite leading coefficient this integer coincides with the total intersection index defined by Edwards in [3, Section 4]. In fact, it can be proven that formula (2.4) reduces to [3, Proposition 4.8, Property (A)].*

3 Variational setting

We use the variational approach to (1.3) as described in [3] and we will stick to the notations of that paper. Given the complex n -dimensional Hermitian space $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$, for any $m \in \mathbb{N}$ let $\mathcal{H}^m := H^m(J, \mathbb{C}^n)$ be the Sobolev space of all H^m -maps from the interval $J := [0, 1]$ into \mathbb{C}^n .

DEFINITION 3.1 *A derivative dependent Hermitian form or a generalized Sturm form, is the form $\Omega(x)[u] = \sum_{i,j=0}^m \langle D^i u(x), \omega_{i,j}(x) D^j u(x) \rangle$, where, each $\omega_{i,j}$ is a smooth path of x -dependent Hermitian matrices with constant leading coefficient $\omega_{m,m} := p_{2m}$.*

We observe that a derivative dependent Hermitian form $\Omega(x)[u]$ actually depends on the m -jet $u, j^m u := (u(x), \dots, u^{(m-1)}(x))$ at the point x and it defines a Hermitian form $q : \mathcal{H}^m \rightarrow \mathbb{R}$ by setting $q(u) := \int_0^1 \Omega(x)[u] dx$. If $v \in \mathcal{H}^m$ and $u \in \mathcal{H}^{2m}$ then, using integration by parts, the corresponding sesquilinear form $q(v, u)$ can be written as

$$q(v, u) = \int_0^1 \langle v(x), l(x, D)u(x) \rangle dx + \phi(v, u) \quad (3.1)$$

where $l(x, D)$ is a differential operator of the form of (1.3) and $\phi(v, u)$ is a sesquilinear form depending only on the $(m-1)$ -jet, $j^{m-1}v(x)$ and on the $(2m-1)$ -jet $j^{2m-1}u(x)$ at the boundary $x = 0, 1$. Thus, there exists a unique linear map $A(x) : \mathbb{C}^{2mn} \rightarrow \mathbb{C}^{mn}$ such that

$$\phi(v, u) = [\langle j^m v(x), A(x) j^{2m} u(x) \rangle]_{x=0}^1. \quad (3.2)$$

The only specific fact that will be needed is that the entries $a_{j, 2m-j-1}$ are all equal to $\pm p_{2m}$. Let $\mathcal{H}_0^m := \mathcal{H}_0^m(J) := \{u \in \mathcal{H}^m : j^m u(0) = 0 = j^m u(1)\}$ and let q_Ω be the restriction of the Hermitian form q to \mathcal{H}_0^m . For each $\lambda \in J$, let us consider the space $\mathcal{H}_0^m([0, \lambda])$ with the form

$\int_{[0,\lambda]} \Omega(x)dx$. Via the substitution $x \mapsto \lambda x$, we transfer this form to $\mathcal{H}_0^m(J)$, so, we come to the forms Ω_λ and q_λ defined respectively by

$$\Omega_\lambda(x)[u] := \sum_{i,j=0}^m \langle D^i u(x), \lambda^{2m-(i+j)} \omega_{i,j}(\lambda x) D^j u(x) \rangle \quad \text{and} \quad q_\lambda(u) := \int_0^1 \Omega_\lambda(x)[u] dx. \quad (3.3)$$

Then $\lambda \mapsto q_\lambda$ is a smooth path of Hermitian forms acting on \mathcal{H}_0^m with $q_1 = q_\Omega$ and with $q_0(u) = \int_J \langle p_{2m} D^m u, D^m u \rangle dx$. Using integration by parts, the sesquilinear Hermitian form $q_\lambda(v, u)$ can be written as

$$q_\lambda(v, u) = \int_0^1 \langle v(x), l_\lambda(x, D) u(x) \rangle dx + \phi_\lambda(v, u),$$

where

$$\phi_\lambda(v, u) := [\langle j^m v(x), A_\lambda(x) j^{2m} u(x) \rangle]_{x=0}^1 \quad \text{and} \quad l_\lambda(x, D) = p_{2m} \frac{d^{2m}}{dx^{2m}} + \sum_{k=0}^{2m-1} p_k(\lambda x) \lambda^{2m-k} \frac{d^k}{dx^k}.$$

DEFINITION 3.2 *A conjugate instant for q_Ω is any point $\lambda \in (0, 1]$ such that $\ker q_\lambda \neq \{0\}$.*

Let C_λ be the path of bounded self-adjoint Fredholm operators associated to q_λ via the Riesz representation theorem.

LEMMA 3.3 *The Hermitian form q_0 is non degenerate. Moreover each q_λ is a Fredholm Hermitian form. (i.e. C_λ is a Fredholm operator). In particular $\dim \ker q_\lambda < +\infty$ and q_λ is non degenerate if and only if $\ker q_\lambda = \{0\}$.*

Proof. That the operator C_0 is an isomorphism can be proven exactly as in [5, Proposition 3.1]. On the other hand each q_λ is a weakly continuous perturbation of q_0 since it differs from q_0 only by derivatives of u of order less than m . Therefore $C_\lambda - C_0$ is compact for all $\lambda \in J$ and hence C_λ is Fredholm of index 0. The last assertion follows from this. \square

When the form q_Ω is non degenerate, i.e. when 1 is not a conjugate instant, according to the definitions and notation of section 2, we introduce the following definition.

DEFINITION 3.4 *The (regularized) Morse index of q_Ω is defined by*

$$\mu_{\text{Mor}}(q_\Omega) := \text{sf}(q_\lambda, J), \quad (3.4)$$

where sf denotes the spectral flow of the path q_λ i.e., the number of positive eigenvalues of C_λ at $\lambda = 0$ which become negative at $\lambda = 1$ minus the number of negative eigenvalues of C_λ which become positive. (See, for instance [5], for a more detailed exposition).

Solution space and EM-index of the boundary value problem.

DEFINITION 3.5 *Let Ω be a derivative dependent Hermitian form on \mathcal{H}^m . $u \in \mathcal{H}^m$ will be called a solution of Ω if it is orthogonal with respect to q to \mathcal{H}_0^m .*

If Σ is the set of all solutions of Ω , by general facts on ODE, it can be proved that Σ is a subspace of \mathcal{H}^m of dimension $2mn$ and

$$h[u] := \Im \langle j^m u(x), A(x) j^{2m} u(x) \rangle$$

is independent of the choice of x . (See [3, Section 1], for further details). Furthermore, h is immediately seen to be a non degenerate Hermitian form on Σ . To prove this fact, we first introduce the map $A^\#$, as follows:

$$A^\#(x) : \Sigma \longrightarrow \mathbb{C}^{nm} \oplus \mathbb{C}^{nm}, \quad \text{by} \quad A^\#(x)[u] := (j^m u(x), A(x) j^{2m} u(x)).$$

Let $A(x) = [a_{jk}(x)]_{j,k}$. From the equalities $a_{j,2m-j-1}(x) = \pm p_{2m}$ and $a_{j,k}(x) = 0$ for $|j+k| \geq 2m$ we have that the matrix $A^\#(x)$ is non singular, hence $A^\#(x)$ is 1-1 and onto. Now the non degeneracy of h follows from the fact that the Hermitian form $(v, w) \mapsto \Im \langle v, w \rangle$ on $\mathbb{C}^{nm} \oplus \mathbb{C}^{nm}$ is non degenerate.

DEFINITION 3.6 *By the solution space of Ω we mean the pair (Σ, h) consisting of the solutions space Σ and the non degenerate Hermitian form h defined on them by*

$$h[u] := \Im \langle j^m u(x), A(x) j^{2m} u(x) \rangle.$$

Before proceeding further we observe that the point $\lambda_0 \in (0, 1)$ is a *conjugate point* if there exists a non trivial solution u of the Dirichlet boundary value problem

$$\begin{cases} l(x, D)u(x) = 0, & \forall x \in [0, \lambda_0] \\ j^m u(0) = 0 = j^m u(\lambda_0). \end{cases} \quad (3.5)$$

Now let $\mathbb{C}^{4mn} = (\mathbb{C}^{mn})^4$, and let \tilde{h} be the Hermitian form given by $\tilde{h}(v_1, w_1, v_2, w_2) := -\Im \langle v_1, w_1 \rangle + \Im \langle v_2, w_2 \rangle$. By an easy calculation obtained by taking into account that superhermitian structure $h[u]$ is independent of x , it follows that the image of the map

$$\tilde{A}^\# := A^\#(0) \oplus A^\#(1) : \Sigma \rightarrow \mathbb{C}^{4mn} : u \mapsto (A^\# u(0), A^\# u(1))$$

is an element of $\mathcal{U}(\mathbb{C}^{4mn}, \tilde{h})$. It is a well-known fact that conjugate points cannot accumulate at 0 and thus we can find an $\varepsilon > 0$ such that there are no conjugate points in the interval $[0, \varepsilon]$.

Now denoting by $a : [\varepsilon, 1] \rightarrow \mathcal{U}(\mathbb{C}^{4mn}, \tilde{h})$ the path defined by $a(\lambda) := \text{Im } \tilde{A}_\lambda^\#$, its EM-index is well-defined and independent on ε . Thus we are entitled to give the following.

DEFINITION 3.7 *Let $P_0 := \{0\} \oplus \mathbb{C}^{mn} \oplus \{0\} \oplus \mathbb{C}^{mn}$. We define the EM-index of Ω as the integer given by*

$$\mu_{\text{EM}}(\Omega) := \mu_{\text{EM}}(a|_{[\varepsilon, 1]}, P_0).$$

4 The main result

THEOREM 1 (Generalized Sturm oscillation theorem). *Under notations above, we have:*

$$\mu_{\text{EM}}(u) = \mu_{\text{Mor}}(q_\Omega).$$

Proof. We split the proof into some steps.

The result holds for regular paths. Let q be the path of Fredholm Hermitian forms defined by

$$q_\lambda(u) := \int_0^1 \sum_{i,j=0}^m \langle D^i u(x), \lambda^{2m-(i+j)} \omega_{i,j}(\lambda x) D^j u(x) \rangle dx. \quad (4.1)$$

In order to prove the thesis, as consequence of propositions 2.2 and 2.9, it is enough to show that at each crossing point λ_0 , we have

$$\text{sign } \Gamma(q, \lambda_0) = \text{sign } \Gamma(a, P_0, \lambda_0).$$

Now let λ_0 be a crossing point and let us denote by \cdot the derivative with respect to λ . Thus we have

$$\begin{aligned} \dot{q}_{\lambda_0}(v, u) &= \left. \frac{d}{d\lambda} \right|_{\lambda=\lambda_0} \int_0^1 \Omega_\lambda(x) [v, u] dx = \left. \frac{d}{d\lambda} \right|_{\lambda=\lambda_0} \left[\int_0^1 \langle v(x), l_\lambda(x, D) u(x) \rangle dx + \phi_\lambda(v, u) \right] \\ &= \int_0^1 \langle v(x), \dot{l}_{\lambda_0}(x, D) u(x) \rangle + \dot{\phi}_{\lambda_0}(v, u). \end{aligned}$$

We set $S(x, D) := \sum_{k=0}^{2m-1} p_k(x) \frac{d^k}{dx^k}$ and $S_\lambda(x, D) = \lambda^{2m-k} S(\lambda x)$. Thus $l_\lambda(x, D)$ can be written as $p_{2m} \frac{d^{2m}}{dx^{2m}} + S_\lambda(x, D)$ and the following result holds.

LEMMA 4.1 *If u is a solution of $l(x, D)u = 0$ then $u_s(x) := u(\frac{s}{\lambda}x)$ is a solution of*

$$p_{2m} \frac{d^{2m}}{dx^{2m}} u_s(x) + S_s(x, D)u_s(x) = 0.$$

Proof. It follows by a straightforward calculations. \square

Therefore for any $s \in (0, 1]$ the function $u_s(x)$ solves the Cauchy problem

$$\begin{cases} p_{2m} \frac{d^{2m}}{dx^{2m}} u_s(x) + S_s(x, D)u_s(x) \equiv l_s(x, D)u_s(x) = 0 \\ u_s(0) = 0, u'_s(0) = c_1 u'(0), \dots, D^{2m-1} u_s(0) = c_{2m-1} u^{(2m-1)}(0) \end{cases} \quad (4.2)$$

where for each $j = 1, \dots, 2m-1$, $c_j = \frac{s^j}{\lambda_0^j}$. Differentiating the Cauchy problem (4.2) with respect to s and evaluating at $s = \lambda_0$, we get

$$\begin{cases} l_{\lambda_0}(x, D)\dot{u}_{\lambda_0}(x) + \dot{S}_{\lambda_0}(x, D)u_{\lambda_0}(x) = 0 \\ \dot{u}_{\lambda_0}(0) = \dots = \dot{u}_{\lambda_0}^{(2m-1)}(0) = 0. \end{cases} \quad (4.3)$$

If $u \in \ker q_{\lambda_0}$, performing integration by parts and observing that $u_{\lambda_0}(\cdot) = u(\cdot)$, as consequence of equation (4.3), we have

$$\begin{aligned} \dot{q}_{\lambda_0}(u) &= \int_0^1 \langle u_{\lambda}(x), \dot{l}_{\lambda_0}(x, D)u_{\lambda}(x) \rangle + \dot{\phi}_{\lambda}(u_{\lambda}) = - \int_0^1 \langle u_{\lambda}(x), l_{\lambda_0}(x, D)\dot{u}_{\lambda}(x) \rangle dx + \dot{\phi}_{\lambda_0}(u_{\lambda}) = \\ &= - \int_0^1 \langle l_{\lambda_0}(x, D)u_{\lambda}(x), \dot{u}_{\lambda}(x) \rangle dx + \dot{\phi}_{\lambda_0}(u_{\lambda}) = \dot{\phi}_{\lambda_0}(u_{\lambda}) = \dot{\phi}_{\lambda_0}(u). \end{aligned}$$

Moreover

$$\begin{aligned} \dot{\phi}_{\lambda_0}(u) &= \frac{d}{d\lambda} \Big|_{\lambda=\lambda_0} \left\{ [\langle j^m u(x), A_{\lambda}(x)j^{2m}u(x) \rangle]_{x=0}^1 \right\} = [\langle j^m u(x), \dot{A}_{\lambda_0}(x)j^{2m}u(x) \rangle]_{x=0}^1 = \\ &= -\langle j^m u(0), \dot{A}_{\lambda_0}(0)j^{2m}u(0) \rangle + \langle j^m u(1), \dot{A}_{\lambda_0}(1)j^{2m}u(1) \rangle. \end{aligned}$$

Since \dot{q}_{λ_0} is a Hermitian form, in particular it is a real-valued function; thus we have

$$\begin{aligned} \dot{q}_{\lambda_0}(u) &= \Re \dot{q}_{\lambda_0}(u) = \Re \dot{\phi}_{\lambda_0}(u) = \Re [\langle j^m u(x), \dot{A}_{\lambda_0}(x)j^{2m}u(x) \rangle]_{x=0}^1 = \\ &= -\Re \langle j^m u(0), \dot{A}_{\lambda_0}(0)j^{2m}u(0) \rangle + \Re \langle j^m u(1), \dot{A}_{\lambda_0}(1)j^{2m}u(1) \rangle. \end{aligned}$$

Since $\Re \langle u, v \rangle = \Im \langle u, -iv \rangle$, we can conclude that

$$\begin{aligned} \dot{q}_{\lambda_0}(u) &= -\Re \langle j^m u(0), \dot{A}_{\lambda_0}(0)j^{2m}u(0) \rangle + \Re \langle j^m u(1), \dot{A}_{\lambda_0}(1)j^{2m}u(1) \rangle = \\ &= -\Im \langle j^m u(0), -i\dot{A}_{\lambda_0}(0)j^{2m}u(0) \rangle + \Im \langle j^m u(1), -i\dot{A}_{\lambda_0}(1)j^{2m}u(1) \rangle = \\ &= -h[j\dot{A}_{\lambda_0}^{\#}(0)[u]] + h[j\dot{A}_{\lambda_0}^{\#}(1)[u]] = \tilde{h}[\tilde{j}\dot{u}^{\#}(\lambda_0)] = \\ &= \Gamma(a, P_0, \lambda_0)(u) \end{aligned}$$

where, for $k = 0, 1$, we denoted by $\dot{A}_{\lambda_0}^{\#}(k)[u]$ the pair $(j^m u(k), \dot{A}_{\lambda_0}(k)j^{2m}u(k))$, by $\dot{u}^{\#}(\lambda_0)$ the element $(j^m u(0), \dot{A}_{\lambda_0}(0)j^{2m}u(0), j^m u(1), \dot{A}_{\lambda_0}(1)j^{2m}u(1)) \in \dot{a}(\lambda_0)$ and where the last equality follows by remark 2.7. The above calculations shown that regular crossings of q correspond to regular crossings of a . Furthermore, the crossing forms at each crossing point associated to the path of Fredholm Hermitian forms and to the path of superlagrangian subspaces are the same and therefore their signatures coincide; in symbols we have

$$\text{sign } \Gamma(q, \lambda_0) = \text{sign } \Gamma(a, P_0, \lambda_0).$$

Now the conclusion of the first step follows by the previous calculations and by summing over all crossings.

Second step. The general case. In order to conclude remains to show that it is possible to extend the above calculation to general paths having not only regular crossings. For each $\lambda \in [0, 1]$ let us consider the closed unbounded Fredholm operator A_λ on $L^2(J, \mathbb{C}^{m_n})$ with domain $\mathcal{D}(A_\lambda) = \{u \in \mathcal{H}^{2m} : j^{m-1}u(0) = 0 = j^{m-1}u(1)\}$ defined by $A_\lambda u := l_\lambda(x, D)u$. By applying a perturbation argument proven in [9, Theorem 4.22] to the path of operators A_λ , we can find a $\delta > 0$ such that $A_\lambda^\delta := A_\lambda + \delta \text{Id}$ is a path of self-adjoint Fredholm operators with only regular crossing points. Let $q_\lambda^\delta(u)$ be the Hermitian form on \mathcal{H}_0^m given by $q_\lambda^\delta(u) := \langle u, A_\lambda^\delta(u) \rangle_{L^2} + \frac{1}{2}\delta \|u\|_{L^2}^2$. By this choice of δ and by applying the first step to the perturbed path q_λ^δ , we conclude the proof. \square

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